Angle of Attack Envelope for a Spinning Re-Entry Body

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An approximate technique for determining the angle of attack envelope for a spinning re-entry body with a large initial angle of attack is developed. The principal restriction is that the aerodynamic torque be proportional to $-\eta(t)\sin\theta$ where θ is the angle of attack and $\eta(t)$ is a continuous function of time determined from a point-mass trajectory. In most cases, $\eta(t)$ is directly proportional to dynamic pressure. In the trivial case when η is a constant, θ oscillates periodically between two bounds. When η is a variable the bounds themselves are functions of time and the loci of the bounds define the angle of attack envelope. A perturbation technique is used to generate the envelope as a function of η. The fact that the envelope can be determined as a function of η rather than time is most significant because each solution (the envelope as a function of η) applies to all point-mass trajectories. The parameter η includes the spin rate. Hence each solution also applies to all spin rates, except zero for which a singularity occurs. Envelopes are generated for the initial condition of pure spin with initial angles of 30°, 90°, and 150°. Comparisons are made with exact solutions for four cases.

Introduction

A SPINNING body re-enters the atmosphere with a large initial angle of attack. The body has mass and aerodynamic symmetry. The atmospheric restoring torque is proportional to the sine of the angle of attack. (Using Newtonian theory this condition is exact for a sphere with mass center displaced from the geometric center and is a good approximation for a slender cone.) The problem is to determine the angle of attack envelope during re-entry.

The corresponding problem for small angles of attack was solved by Leon.[1] He assumed that the linear velocity is constant during the time of interest, that the atmospheric density is exponential with altitude, and that aerodynamic damping is ignorable. Because of the restriction to small angles, Leon was able to linearize the equations of motion and obtain a solution in terms of Bessel functions.

Murphy[2] has investigated the case of a nonlinear restoring torque (a cubic rather than a sine function of the angle of attack). Using a perturbation technique to determine the effects of atmospheric density variations, Murphy obtained two first order differential equations which, when integrated, provide the angle of attack envelope. However, because of an assumption regarding the cosine of the angle of attack, the results are inaccurate for spinning bodies with large angles of attack.

A different formulation than Leon's and Murphy's is used in this paper to handle the case of large angles of attack. The envelope is obtained directly with no attempt to describe the instantaneous motion within the envelope. The direction, but not the magnitude, of the linear velocity vector is assumed to be constant. No restriction is placed on the variation of atmospheric density with altitude. Aerodynamic damping is ignored.

The paper is arranged as follows. First the exact equations of motion are reduced to second order. Numerical integration of the reduced system is unattractive because of the high frequency of the angle of attack θ. Only in the trivial case when the torque is not an explicit function of time can an analytic solution be obtained. This solution, which is summarized in the next section, serves as the zero order approximation to the nontrivial case. The approximate method of solution for the nontrivial case is presented in the following section. This method still requires the numerical integration of a first order differential equation and the simultaneous root extraction of a cubic equation. However, the high frequency content of θ is "averaged out" and the numerical integration can be performed by hand. In the next section a simplified approach, which allows the integration to be performed independently of the root extraction under a certain condition, is developed. Envelopes for the initial condition of pure spin with three initial angles of attack are generated in the following section using the simplified approach. Next the region of validity for these solutions is established. Comparisons with four exact solutions, obtained by integrating the complete equations, are made in the last section.

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Equations of Motion

XYZ is a nonrotating coordinate system with origin at the mass center of the body. The Z axis points in the direction of motion of the mass center. A second axis system xyz, also with origin at the mass center, is fixed to the body with z along the axis of symmetry. The angle of attack θ is the angle between z and Z. The transformation between xyz and XYZ is defined in terms of the Euler angles ϕ, θ, and ψ (Fig. 1) as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad (1)$$

The problem is to describe the motion of the body about the mass center when acted upon by a torque

$$Q = -p(t)\sin\theta \quad (2)$$

about the line of nodes (the N axis in Fig. 1) where $p(t)$ is any continuous function of time. The body is assumed to be aerodynamically stable in which case $p(t)$ is a positive function. Q is derivable from a scalar potential function V in the manner

$$Q = -\partial V/\partial\theta \qquad V = -p(t)\cos\theta \quad (3)$$

The kinetic energy of rotation about the mass center is given by

$$T = \tfrac{1}{2}A(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \tfrac{1}{2}C(\dot\psi + \dot\phi\cos\theta)^2 \quad (4)$$

where A is the moment of inertia about x and y, and C is the moment of inertia about z. Dots denote total derivatives with respect to time.

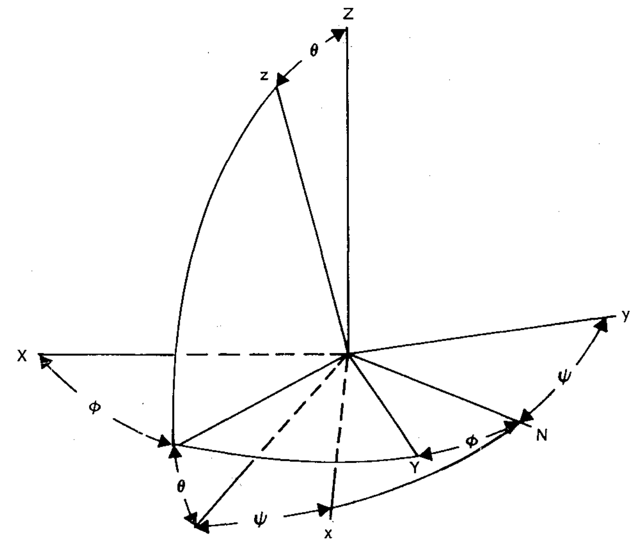

Fig. 1 The Euler angles ϕ, θ, and ψ.

Since T and V are not explicit functions of ψ and ϕ, there exist two momentum integrals:

$$\partial T/\partial\dot{\psi} = C(\dot{\psi} + \dot{\phi}\cos\theta) = \text{Cs} \tag{5}$$

$$\partial T/\partial\dot{\phi} = A\dot{\phi}\sin^2\theta + \text{Cs}\cos\theta = \text{Ab} \tag{6}$$

where s and b are constants. The spin rate s is the component of angular velocity along z, and Ab is the component of angular momentum along Z.

The energy $H = T + V$ is not constant if V is an explicit function of time. The instantaneous value for H satisfies the equation

$$\dot{H} = \partial V/\partial t = -\dot{p}\cos\theta \tag{7}$$

Using (5) and (6) to eliminate $\dot{\phi}$ and $\dot{\psi}$, H can be written as an explicit function of θ, ψ, and t:

$$H = \frac{1}{2}A\dot{\theta}^2 + \frac{(Ab - Cs\cos\theta)^2}{2A\sin^2\theta} + \frac{1}{2}Cs^2 - p(t)\cos\theta \tag{8}$$

With $a = Cs/A$, $\zeta = b/a$, $\tau = at$, $u = \cos\theta$, $\xi = 2H/(a^2A) - A/C$, $\eta = 2p/(a^2A)$, (7) and (8) can be written in the nondimensional forms

$$d\xi/d\eta = -u \tag{9}$$

$$(du/d\tau)^2 = (1 - u^2)(\xi + \eta u) - (\zeta - u)^2 \tag{10}$$

Thus the system can be reduced to two first order differential equations. Further reduction does not appear to be possible, except when η is a constant.

If u can be determined as a function of τ, the instantaneous values for $d\phi/d\tau$ and $d\psi/d\tau$ can be evaluated from

$$d\phi/d\tau = (\zeta - u)/(1 - u^2) \tag{11}$$

$$d\psi/d\tau = (A/C) - u(\zeta - u)/(1 - u^2) \tag{12}$$

and ϕ and ψ can be reduced to quadratures.

Solution for Constant η

When η is a constant, the problem is equivalent to the Lagrange top.[3] From spinning shell theory, the motion is stable about $\theta = 0$ when $\eta > -\frac{1}{2}$. The parameter ξ is also constant, and u oscillates periodically between two roots of the cubic equation

$$f(u) \equiv (1 - u^2)(\xi + \eta u) - (\zeta - u)^2 = 0 \tag{13}$$

obtained by setting $du/d\tau$ equal to zero in (10). For large positive (negative) u, $f(u)$ is negative (positive). For $u = \pm 1$, $f(u) = -(\zeta - u)^2 \leq 0$. Hence, at least one root, say u_1, is less than or equal to -1. Since $u = \cos\theta$, the region of interest is $-1 \leq u \leq 1$. Physical motion can occur only when $du/d\tau$ is real, which requires $f(u) > 0$. Therefore, $f(u)$ must appear as in Fig. 2 with the two remaining roots, u_2 and u_3, between ± 1. The roots are ordered as $u_1 \leq -1 \leq u_2 \leq u_3 \leq 1$ with physical motion in the interval $u_2 \leq u \leq u_3$.

Equation (10) can be written as

$$(du/d\tau)^2 = \eta(u - u_1)(u - u_2)(u_3 - u) \tag{14}$$

which has the solution

$$u = u_3 - (u_3 - u_2)\,sn^2\,[n(\tau - \tau_0), k] \tag{15}$$

where τ_0 is the constant of integration and sn is the Jacobian elliptic function with

$$n = \tfrac{1}{2}\eta^{1/2}(u_3 - u_1)^{1/2} \qquad k = (u_3 - u_2)^{1/2}(u_3 - u_1)^{-1/2} = \text{modulus}$$

Hence u is periodic in τ with period $2K/n$ where

$$K = \int_0^{\pi/2}(1 - k^2\sin^2\lambda)^{-1/2}d\lambda \tag{16}$$

is the complete elliptic integral of the first kind.

Solution for Variable η

When η is a variable the system is represented by (9) and (10). If one were able to solve these equations for given initial conditions, the roots u_i could be determined at each point along the trajectory. The angle of attack envelope is the locus of $\theta_2 = \cos^{-1}u_2$ and $\theta_3 = \cos^{-1}u_3$. For convenience the locus of u_2 and u_3 will also be called the angle of attack envelope. An analytic solution to (9) and (10) appears hopeless, even for simple functions $\eta(\tau)$. On the other hand, a numerical solution of (9) and (10) would be inefficient if one only wanted to determine the angle of attack envelope. The reason is that u_2 and u_3 may change slowly even when the frequency of u is high.

An approximate method, essentially a perturbation technique, for obtaining the angle of attack envelope is described below. Let u^* denote the average value for u during any one cycle in u. If the roots u_i change sufficiently slowly, u^* can be generated from (15) in the manner

$$\begin{aligned} u^* &= u_3 - (u_3 - u_2)\frac{1}{K}\int_0^K sn^2\alpha\,d\alpha \\ &= u_3 - (u_3 - u_2)(K - E)/(k^2K) \end{aligned} \tag{17}$$

where

$$E = \int_0^{\pi/2}(1 - k^2\sin^2\lambda)^{1/2}d\lambda \tag{18}$$

is the complete elliptic integral of the second kind. The average u^* is uniquely determined by the u_i which, in turn, are uniquely determined by ξ, η, and the constant ζ. The cubic invariants

$$\begin{aligned} -\eta(u_1 + u_2 + u_3) &= \xi + 1 \\ -\eta(u_1u_2 + u_2u_3 + u_3u_1) &= \eta + 2\zeta \\ -\eta(u_1u_2u_3) &= \zeta^2 - \xi \end{aligned} \tag{19}$$

which relate the roots u_i and the set $\xi\eta\zeta$, are determined by setting the right-hand sides of (10) and (14) equal to zero and equating coefficients of like powers of u.

The variables ξ and η are not mutually independent, since they are related by (9), and (9) depends on u. The main idea of the proposed method is to replace u by u^* so that (9) becomes

$$d\xi/d\eta = -u^* \tag{20}$$

The angle of attack envelope is obtained by numerically integrating (20) and solving for the roots to (13) at discrete values of η. The right hand side of (20) is an explicit function of the roots u_i. In order to integrate (20) it is necessary to express the roots (at each value of η) in terms of ξ and η. Consequently the integration of (20) and the root determination must proceed simultaneously.

Comments on the Method of Solution

In the proposed method for determining the angle of attack envelope, the integration of (20) replaces the simultaneous integration of (9) and (10). No attempt is made to determine the instantaneous value of u. Rather the "average value" u^* is used in (20) to determine the relation between ξ and η. Consequently, the integration of (20) is unaffected by the frequency of u and can be easily performed by hand calculation using relatively large intervals in η.

The use of u^* implies that the changes in the roots are so small that they can be treated as constants over any one cycle in u. However, the criterion for the validity of (20) can be more simply stated; the change in u^* during any one cycle in u must be small compared to 1. Thus the solutions (the envelopes as functions of η) are independent of the frequency of u only for frequencies sufficiently high. To determine if this condition is satisfied, it is necessary to examine the actual relation between η and τ. This is done in a later section for a particular case of interest. Suffice it to say that the method breaks down as the spin rate approaches zero and as the entry velocity approaches infinity.

In developing the nondimensional form for the equations of motion, the spin rate was incorporated in the parameter η and appears nowhere else in the formulation. This fact is of considerable importance because each solution for a given set of initial conditions applies to all spin rates, except zero for which a singularity occurs. However, the accuracy of the solution deteriorates as the spin rate approaches zero.

Of greatest significance is that the envelope can be generated as a function of η independent of the relation between η and the nondimensional time τ. The function $\eta(\tau)$ is determined by the particular point-mass trajectory. Hence each solution for a given set of initial conditions applies to all point-mass trajectories.

In simple cases, η is directly proportional to dynamic pressure. Suppose that the torque is given by

$$Q = -qSdC_N \qquad C_N = \kappa \sin\theta$$

where q is the dynamic pressure, S is the reference area, d is the distance between the centers of pressure and mass, and C_N is the normal force coefficient. If d and κ are constants, η varies linearly with q:

$$\eta/q = 2\kappa Sd/(a^2A) = \text{const} \tag{21}$$

Since q is determined by the motion of the mass center, η is likewise determined and the angle of attack envelope can be expressed as a function of q, time, altitude, or any other appropriate parameter. Even if d and κ were to vary with Mach number and Reynold's number, η would be determined by the motion of the mass center.

As a first approximation one might determine the motion of the mass center using a drag function for zero angle of attack. However, the motion of the mass center also depends on the angle of attack, and in some cases this dependence cannot be ignored. In these cases a table of u^* (obtained by the method of this paper) for discrete values of η could be "read into" a point-mass trajectory program. At each integration step η would be calculated, the average angle of attack corresponding to u^* would be interpolated from the table, and the drag force for the average angle of attack would be used to determine the acceleration. Thus one can determine directly the motion of the mass center, taking into account the change in drag due to angle of attack.

A Simplified Form

As stated before, the integration of (20) and the root determination of (13) must be performed simultaneously. This is an inconvenience if the calculations are to be performed by hand, or if knowledge of the roots is necessary at only a few values of η.

If the condition

$$l^2 = |2(u_3 - u_2)/(u_2 + u_3 - 2u_1)| \ll 1 \tag{22}$$

is satisfied, and if terms of order l^2 and higher are dropped, (20) can be approximated by

$$\frac{d\xi}{d\eta} = -v$$

$$v = -\frac{1}{3\eta}\left\{1 + \xi - \left[(1 + \xi)^2 + 3\eta(\eta + 2\zeta)\right]^{1/2}\right\} \tag{23}$$

Since v is an explicit function of ξ and η, the integration of (23) can proceed without knowledge of the roots. Hence the roots need not be determined until after ξ has been generated as a function of η.

The proof for (23) follows. Since $0 \leq k^2 \leq l^2$, the integrands in (16) and (18) can be expanded in powers of $k^2 \sin^2\lambda$ and integrated. One then obtains

$$K = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot3}{2\cdot4}\right)^2 k^4 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 k^6 + \ldots\right]$$

$$E = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1\cdot3}{2\cdot4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 \frac{k^6}{5} - \ldots\right]$$

$$\frac{K - E}{k^2K} = \frac{1}{2} + \frac{1}{16}k^2 + \frac{1}{32}k^4 + \ldots$$

and finally

$$u^* = \tfrac{1}{2}(u_3 + u_2) - (u_3 - u_2)\left[\tfrac{1}{16}k^2 + \tfrac{1}{32}k^4 + \ldots\right] \tag{24}$$

Substituting (19) into the expression for v one obtains

$$v = \tfrac{1}{3}(u_1 + u_2 + u_3) + \tfrac{1}{3}(u_1^2 + u_2^2 + u_3^2 - u_1u_2 - u_2u_3 - u_3u_1)^{1/2} \tag{25}$$

$$= \tfrac{1}{3}(u_1 + u_2 + u_3) + \tfrac{1}{6}(u_3 + u_2 - 2u_1)(1 + \tfrac{3}{4}l^4)^{1/2}$$

When terms of order l^2 and higher are ignored, (24) and (25) reduce to

$$v = u^* = \tfrac{1}{2}(u_3 + u_2) \tag{26}$$

which completes the proof. Observe that, for $l^2 \ll 1$, v is the average value of u_3 and u_2.

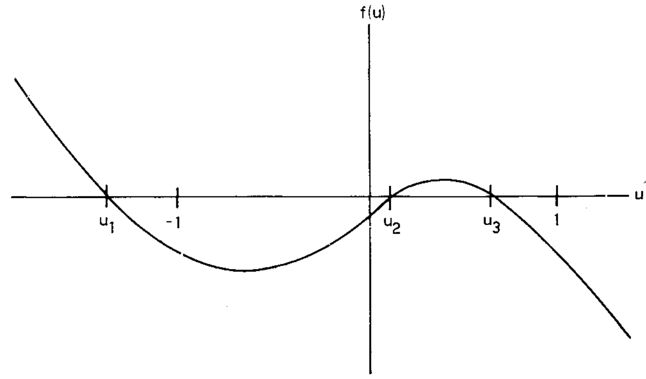

Fig. 2 The region of motion, $u_2 \leq u \leq u_3$.

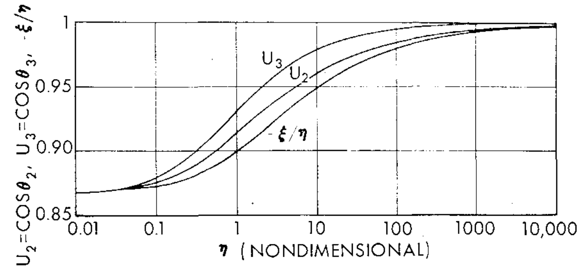

Fig. 3 The envelope and $-\xi/\eta$ for θ_0, = 30° (u_0 = 0.86603).

Envelopes for Initial Conditions of Pure Spin

Initial conditions of pure spin with nonzero angles of attack are of first interest in re-entry dynamics. Pure spin means that the z axis is motionless. Let u_0, u_0', ϕ_0', and ξ_0 denote the values of u, $du/d\tau$, $d\phi/d\tau$, and ξ at the start of re-entry; that is, when $\eta = 0$. For the stated conditions one has $u_0 \neq 1$ and $u_0' = \phi_0' = 0$. The values $\zeta = u_0$ and $\xi_0 = 0$ are determined by evaluating (11) and (10), respectively, at $\eta = 0$.

It was assumed and subsequently confirmed that $l^2 \ll 1$ for the foregoing initial conditions. The cases considered were $u_0 = 0.86603$, 0, and -0.86603 which correspond to the initial angles $\theta_0 = 30°$, 90°, and 150°, respectively. The angle of attack envelopes were generated for these cases by first numerically integrating (23) to obtain ξ for discrete values of η and then extracting the roots to (13). The roots u_2 and u_3, which define the envelope for u are graphed as functions of η in Figs. 3–5 for the three initial angles. The parameter ξ, which has the same form as u_2 and u_3 when normalized on $-\eta$, is also plotted in these figures. The average value v is not shown since it is almost half the distance between u_2 and u_3, the error being of order l^4. The maximum values for l^2 were found to be 0.016, 0.097, and 0.220 for Figs. 3–5, respectively. Since l^2 is small compared to 1, the use of (23) is justified.

A Necessary Condition

The solutions obtained by the proposed method require that the change in u^* during any one cycle in u be small compared to 1. For the solutions portrayed in Figs. 3–5, this condition is $\Delta v \ll 1$ where Δv is the change in v during any cycle in u. Taking the total derivative of v with respect to η and setting $\zeta = u_0$ (the initial condition of pure spin), one obtains

$$dv/d\eta = -v\partial v/\partial\xi + \partial v/\partial\eta = g/\eta \qquad (27)$$

where

$$g = \frac{\eta + \zeta}{h} - \frac{2}{3}v - \frac{v(1+\xi)}{3h}$$

$$h = [(1+\xi)^2 + 3\eta(\eta + 2\zeta)]^{1/2}$$

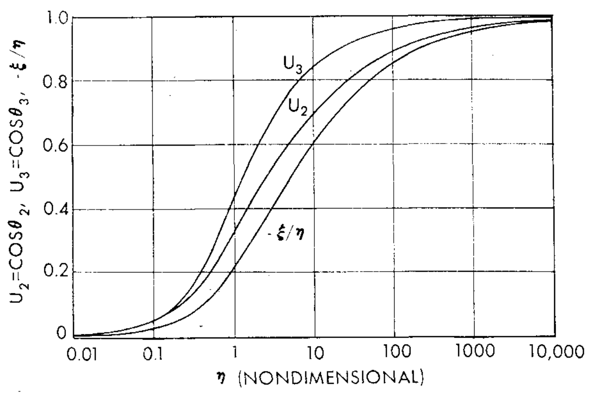

Fig. 4 The envelope and $-\xi/\eta$ for θ_0 = 90° (u_0 = 0).

Let $\Delta\eta$ denote the interval in η during a cycle in u. Then in the limit as $\Delta\eta$ approaches zero, $\Delta v/\Delta\eta$ approaches g/η. The condition $\Delta v \ll 1$ becomes

$$(g/\eta)\Delta\eta \ll 1 \qquad (28)$$

To determine $\Delta\eta$ it is necessary to examine the particular function $\eta(\tau)$. Thus while the solutions themselves are independent of the relation between η and τ, this relation must be introduced in order to establish the region of validity for the solutions. As an example, the function $\eta(\tau)$ consistent with the conditions imposed by Leon[1] (namely, constant linear velocity, an exponential atmosphere, and a constant moment coefficient) will be used. For these conditions $\eta(\tau)$ satisfies the differential equation

$$d\eta/d\tau = c\eta \qquad c = \beta w \sin\gamma/a \qquad (29)$$

where w is the linear velocity, γ is the entry angle measured from the local horizontal, and β is the atmospheric scale constant (for the terrestrial atmosphere $\beta^{-1} \approx 22{,}000$ ft). Let $\Delta\tau$ denote the period of oscillation for u. In the limit as $\Delta\tau$ approaches zero, (29) provides $\Delta\eta = c\eta\Delta\tau$. For $k^2 \ll 1$ (which is no more stringent than $l^2 \ll 1$) $\Delta\tau = \pi/n$. Substituting $\Delta\eta$ and $\Delta\tau$ into (28) one obtains the necessary condition

$$\pi cg/n \ll 1 \qquad (30)$$

which is to be satisfied for all η. Obviously (30) is not satisfied if the spin rate is zero, or if the entry velocity is infinite. The inequality (30) provides quantitative meaning to the earlier statement that the solutions are independent of the frequency of u only for frequencies sufficiently high.

Comparison with Exact Solutions

A comparison between the predicted angle of attack envelope (for $\theta_0 = 42°$) and exact values for θ obtained from numerical solutions for four cases by D. Graham are shown in Fig. 6. The solid curves labeled θ_2 and θ_3, which define the predicted angle of attack envelope, were generated using the proposed method. The discrete points in Fig. 6 were obtained by integrating the complete equations on a digital computer.

Graham's four runs are detailed below. The body was a slender cone for which the restoring torque varied approximately as $\sin\theta$. The physical data was: $C = 0.05$ slug-ft^2, $A = 0.8$ slug-ft^2, $Sd = 0.33$ ft^3, and $\kappa = 2$. The entry velocity w was 23,000 ft-sec^{-1}, the entry angle γ measured from the local horizontal was 22°, and the initial angle of attack θ_0 was 42°. The program was initiated at 500,000 ft and the 1959 ARDC atmosphere was used. The runs differed only in the spin rate, e.g., s = 120, 80, 40, and 20 rpm. Since the computer printed out Q and θ, it was possible to calculate η

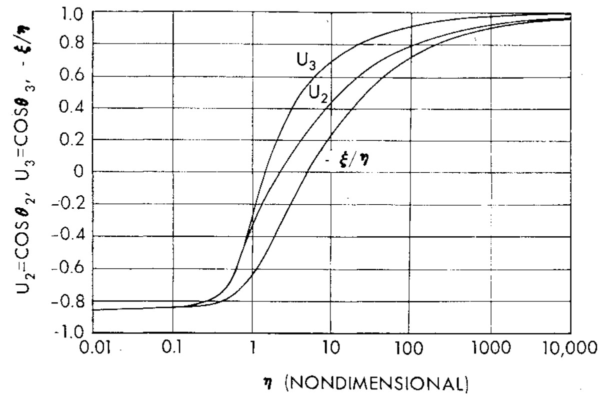

Fig. 5 The envelope and $-\xi/\eta$ for θ_0 = 150° (u_0 = −0.86603).

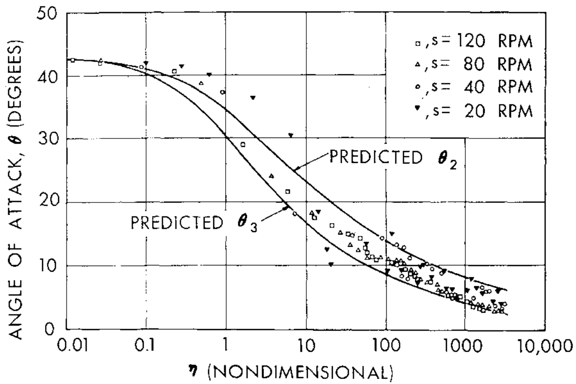

Fig. 6 Comparison between the predicted envelope and exact values of $\theta(\theta_0 = 42°)$.

from the expression $\eta = -2Q(a^2A \sin\theta)^{-1}$ for each value of θ. These results are plotted in Fig. 6.

Ignore temporarily the solid curves in Fig. 6. A striking feature of this figure is that the discrete points lie in roughly the same band for all four spin rates. This would not have been the case had θ been plotted as a function of time, altitude, dynamic pressure, or other more common variable. What is implied is that a single envelope of oscillation for all spin rates can be determined empirically by simply plotting θ as a function of η. Furthermore, other parameters than s could be varied, such as S, d, κ, w, γ, C, and A, and the discrete values for θ would all lie in roughly the same band for the given initial conditions provided that the ordinate was η. This points up the fact that the most important contribution of this paper is the use of η as the independent parameter.

Figure 6 shows that the predicted envelope (solid curves) decreases too fast in the region of small η but accurately bounds θ for large η. Furthermore, the predicted envelope is more accurate for small η when the spin rate s is large. The explanation is that condition (30), which depends on s, is not satisfied in the region of small η. Using Graham's data one finds that $\pi cg/n$ exceeds 10^{-1} in the regions; $\eta < 1.7$ when $s = 120$ rpm, $\eta < 2.8$ when $s = 80$ rpm, $\eta < 6.5$ when $s = 40$ rpm, and $\eta < 20$ when $s = 20$ rpm. Since the change in θ is already significant when $\eta = 1$, the predicted envelope should not be expected to give accurate results for any of Graham's runs. Nevertheless the predicted envelope does bound θ when η exceeds the forestated critical values, in spite of errors in the region of small η. It appears that condition (30) is somewhat stringent, and that usable results can even be obtained when $\pi cg/n \approx 1$.

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